

The Second Variational Formula For the Functional $\int v^{(6)}(g)dV_g$

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Abstract

In this note, we compute the second variational formula for the functional $\int_M v^{(6)}(g)dv_g$, which was introduced by Graham-Juhl [GJ] and the first variational formula was obtained by Chang-Fang ([CF]). We also prove that Einstein manifolds (with dimension ≥ 7) with positive scalar curvature is a strict local maximum within its conformal class, unless the manifold is isometric to round sphere with the standard metric up to a multiple of constant. Note that when (M, g) is locally conformally flat, this functional reduces to the well-studied $\int_M \sigma_3(g)dv_g$. Hence, our result generalize a previous result of Jeff Viaclovsky ([V]) without the locally conformally flat restraint.

Key words and phrases: second variation, renormalized volume coefficients, Bach tensor, Einstein metric.

1 Introduction

In the following, we let (M^n, g) denote a compact, connected, smooth Riemannian manifold without boundary. We denote the Ricci curvature and scalar curvature by Ric and R , respectively. Recall that the Schouten tensor P_{ij} is defined by

$$P_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{R}{2(n-1)} g_{ij} \right),$$

and the Riemann curvature tensor can be written by

$$Riem = W + P \odot g,$$

where W is the Weyl curvature and \odot is the Kulkarni-Nomizu product, which is defined by

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ik}\beta_{jl} + \alpha_{jl}\beta_{ik} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il}, \quad \forall \text{ symmetric 2-tensors } \alpha, \beta.$$

The $\sigma_k(g)$ curvature is defined to be the k -th elementary symmetric polynomial of the eigenvalue of the Schouten tensor P . In [V], Viaclovsky started study of the variational problems of the functional $\int_M \sigma_k(g)dv_g$, he proved that the first variation of the functional $\int_M \sigma_k(g)dv_g$ ($k = 1, 2$) within a conformal class subject to the constraint $Vol(M, g) = 1$ is a metric satisfying $\sigma_k(g) \equiv \text{const}$, and if $k \geq 3$ and the Riemannian manifold is locally conformally flat, the same result follows. However, for $k \geq 3$ and the manifold is not locally conformally flat, $\sigma_k(g) \equiv \text{const}$ is not Euler-Lagrange equation of the functional $\int_M \sigma_k dv$ within a conformal class subject to the constraint $Vol(M, g) = 1$.

The renormalized volume coefficients of g , denoted here by $v^{(2k)}(g)$, arose in the late 90s in the physics literature. They are defined in terms of the expansion of the ambient or Poincare

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metric associated to g . If the Riemannian manifold is locally conformally flat, these quantities coincide with the $\sigma_k(g)$ up to a constant. More precisely, it is known that (see [GJ], [CF], or [GHL])

$$\begin{aligned} v^{(2)}(g) &= -\frac{1}{2}\sigma_1(g), & v^{(4)}(g) &= \frac{1}{4}\sigma_2(g), \\ v^{(6)}(g) &= -\frac{1}{8}[\sigma_3(g) + \frac{1}{3(n-4)}(P_g)^{ij}(B_g)_{ij}], \end{aligned}$$

where

$$(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj} \quad (1.1)$$

is the *Bach* tensor of the metric. Just as $\int_M \sigma_k(g^{-1} \circ A_g) dv_g$ is conformally invariant when $2k = n$ and (M, g) is locally conformally flat, Graham showed in [G] that $\int_M v^{(2k)}(g) dv_g$ is also conformally invariant on a general manifold when $2k = n$. Chang and Fang showed in [CF] that, for $n \neq 2k$, the Euler-Lagrange equations for the functional $\int_M v^{(2k)}(g) dv_g$ under conformal variations subject to the constraint $Vol_g(M) = 1$ satisfies $v^{(2k)}(g) = \text{const.}$, which is a generalized characterization for the curvatures $\sigma_k(g^{-1} \circ A_g)$ when (M, g) is locally conformally flat, as given by Viaclovsky [V].

We note that Graham [G] also gives an explicit expression of $v^{(8)}(g)$, but the explicit expression of $v^{(2k)}(g)$ for general k is not known because they are algebraically complicated (see page 1958 of [G]). Thus the study of the $v^{(2k)}(g)$ curvatures involves significant challenges not shared by that of $\sigma_k(g)$: firstly, for $k \geq 3$, $v^{(2k)}(g)$ depends on derivatives of curvature of g —in fact, for $k \geq 3$, $v^{(2k)}(g)$ depends on derivatives of curvatures of order up to $2k - 4$; secondly, the $v^{(2k)}(g)$ are defined via an indirect highly nonlinear inductive algorithm (see [G]). We aim to study the stability of the critical metric of the functional

$$\mathcal{F}_3[g] = \frac{\int_M v^{(6)}(g) dv_g}{(\int_M dv_g)^{(n-6)/n}},$$

within a conformal class. First we recall the theorem of Chang-Fang [CF] (also see Graham [G]).

Theorem 1.1. ([CF]) *Let (M^n, g) be an n -dimensional ($n \geq 7$) compact Riemannian manifold, then the functional $\mathcal{F}_3(g)$ is variational within the conformal class, i.e. the critical metric in $[g]$ satisfies the equation*

$$v^{(6)} \equiv \text{const.} \quad (1.2)$$

If $n = 6$, $\mathcal{F}_3[g]$ is a constant in the conformal class $[g]$.

In this note, we compute the second variational formula of $\mathcal{F}_3[g]$ within its conformal class $[g]$. Our results are

Theorem 1.2. *Let (M^n, g) be an n -dimensional ($n \geq 7$) compact Riemannian manifold with $v^{(6)}(g) = \text{const.}$, then the second variational formula of the functional $\mathcal{F}_3[g]$ within its conformal class at g is*

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3[g_t] = (n-6)V^{-(n-6)/n} \left\{ \int \left[-6v^{(6)}(g)\bar{\phi}^2 + \left(\frac{B_{ij}\bar{\phi}_{ij}}{24(n-4)} + \frac{1}{8}T_{2ij}\bar{\phi}_{ij} - \frac{1}{12}P_{ij}C_{ijk}\bar{\phi}_k \right) \bar{\phi} \right] dv \right\}$$

where $g_t = e^{2u_t}g$, $\frac{\partial}{\partial t} \Big|_{t=0} u_t = \phi$, and $\bar{\phi} = \phi - \frac{\int_M \phi dv_g}{\int_M dv_g}$, T_{2ij} and C_{ijk} are defined in section 2.

Theorem 1.3. *Let (M^n, g) be an n -dimensional ($n \geq 7$) compact Einstein manifold with positive scalar curvature. Then it is a strict local maximum within its conformal class $[g]$, unless (M^n, g) is isometric to S^n with the standard metric up to a multiple of constant.*

Remark 1.1. When (M^n, g) is a locally conformally flat, $v^{(6)}(g) = -\frac{1}{8}\sigma_3(g)$, for the functional $\int_M \sigma_3(g) dv_g$, J. Viaclovsky ([V]) proved that a positive constant sectional curvature metric is a strict local minimum, unless the manifold is isometric to S^n with the standard metric. Our result coincides with his at the locally conformally flat Einstein metrics, however, ours does not need the locally conformally flat assumption.

2 Preliminaries

Let (M^n, g) be an n -dimensional compact Riemannian manifold. Throughout this note, we make the convention that repeated index means summation over 1 to n . First we recall the transformation law of various curvatures under conformal change of metrics. Let $\tilde{g} = e^{2u}g$, $u \in C^\infty(M)$, then the Riemannian curvature tensors satisfy

$$\text{Riem}(\tilde{g}) = e^{2u}(\text{Riem}(g) - \alpha \odot g),$$

where $\alpha_{ij} = u_{ij} - u_i u_j + \frac{|\nabla u|^2}{2} g_{ij}$ (note that u_{ij} means the covariant derivative with respect to the fixed metric g). By contracting, we see that the Ricci curvature and scalar curvature satisfy

$$R_{ij}(\tilde{g}) = R_{ij} - (n-2)\alpha_{ij} - \left(\sum_k \alpha_{kk}\right)g_{ij}, \quad R(\tilde{g}) = e^{-2u}R - 2(n-1)e^{-2u}\sum_k \alpha_{kk}. \quad (2.1)$$

From (2.1) and the definition of Schouten tensor, we see that

$$\tilde{P}_{ij} = P_{ij} - \alpha_{ij}, \quad (2.2)$$

where we denote $P(\tilde{g})$ by \tilde{P} for notations convenience.

Lemma 2.1. *We have the following formulae (see e.g. [GHL])*

- (1) $\nabla^i W_{ijkl} = -(n-3)C_{jkl}$, C_{ijk} is the Cotton tensor defined by $P_{ij,k} - P_{ik,j}$;
- (2) $\nabla^j B_{ij} = (n-4)\sum_{k,l} P_{kl}C_{kli}$;
- (3) $B_{ij} = B_{ji}$, where B_{ij} is defined by (1.1).

The proof of Lemma 2.1 is a direct calculation and one can find it in [GHL].

Let V be a vector space, $A : V \rightarrow V$ a linear map. Define the Newton transformation $T_k(A) : V \rightarrow V$ by:

$$T_k(A) := \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k = \sum_{i=0}^k \sigma_{k-i}(A)(-1)^i A^i,$$

where I is the identity map and $\sigma_k(A)$ is the k -th elementary symmetric polynomial of the eigenvalues of A . Under an orthonormal basis of V , T_k can be written as follows:

$$T_{kij} = \frac{1}{k!} \delta_{i_1 \dots i_k i}^{j_1 \dots j_k j} A_{i_1 j_1} \cdots A_{i_k j_k},$$

where $\delta_{i_1 \dots i_k i}^{j_1 \dots j_k j}$ is the generalized Kronecker notation. We recall some well-known results in this respect, which we will need in our later arguments.

Lemma 2.2. *The Newton transformations T_k satisfy ([R], [GHL])*

- (1) *Newton's formula: $(k+1)\sigma_{k+1}(A) = \text{tr}(T_k A)$;*
- (2) *$\frac{d}{dt}\sigma_k(A_t) = \text{tr}(T_{k-1}\frac{d}{dt}A_t)$, for any family of transformations $A_t : V \rightarrow V$.*
- (3) *$\text{tr}(T_k) = (n-k)\sigma_k(A)$.*

In the following we denote $T_k(g^{-1} \circ P)$ simply by T_k . We have the following formula, which is a direct calculation (see [GHL])

$$\sum_i T_{2ij,i} = - \sum_{k,l} P_{kl} C_{klj}. \quad (2.3)$$

3 The first variational formula and proof of Theorem 1.1

In this section, we will compute the Euler-Lagrange equation for the functional $\mathcal{F}_3(g)$ within the conformal class. For convenience we denote the numerator of $\mathcal{F}_3(g)$ by

$$F(g) = \int_M v^{(6)}(g) dv_g.$$

Under the conformal change of metrics $g_t = e^{2u(t)}g$, by use of (2.2), we see that in local coordinates (see [CF])

$$\begin{aligned} \tilde{P}_i^j &= e^{-2u(t)}(P_i^j - \alpha_i^j) \\ \tilde{B}_i^j &= e^{-4u(t)}\left(B_i^j + (n-4)u^k(g^{jl}C_{ilk} + g^{jl}C_{lik}) + (n-4)u^k u^l g^{pj}W_{ikpl}\right), \end{aligned}$$

where we write $\alpha_{ij} = u_{ij}(t) - u_i(t)u_j(t) + \frac{1}{2}|\nabla_g u(t)|^2 g_{ij}$ and we make the convention that $\tilde{P}_{ij} = P_{ij}(g_t)$, $\tilde{B}_{ij} = B_{ij}(g_t)$, etc, for notations convenience.

For notions convenience we denote $\frac{d}{dt}$ by δ . Denote $\frac{\partial}{\partial t}\Big|_{t=0} u = \phi$, and $\frac{\partial^2}{\partial t^2}\Big|_{t=0} u = \psi$. With the above preparations, we have

$$\begin{aligned} \delta \tilde{P}_i^j &= -2(\delta u)\tilde{P}_i^j - e^{-2u}\delta\alpha_i^j \\ \delta \tilde{B}_i^j &= -4(\delta u)\tilde{B}_i^j + (n-4)e^{-4u}\left((\delta u)^k(g^{jl}C_{ilk} + g^{jl}C_{lik}) \right. \\ &\quad \left. + (\delta u)^k u^l g^{pj}W_{ikpl} + u^k(\delta u)^l g^{pj}W_{ikpl}\right) \\ \delta(dv_{g_t}) &= n(\delta u)dv_{g_t}. \end{aligned}$$

Now we derive the first variation formula for \mathcal{F}_3 . First we have

$$-8\delta F = \int \delta(\sigma_3(g_t)dv_{g_t}) + \frac{1}{3(n-4)}\delta(\tilde{P}_i^j \tilde{B}_j^i dv_{g_t}). \quad (3.1)$$

By use of Lemma 2.2, we have

$$\begin{aligned} \delta(\sigma_3(g_t)) &= \tilde{T}_{2j}^i \delta \tilde{P}_i^j = \tilde{T}_{2j}^i (-2(\delta u)\tilde{P}_i^j - e^{-2u}\delta\alpha_i^j) \\ &= -6(\delta u)\sigma_3(g_t) - e^{-2u}\tilde{T}_{2j}^i \delta\alpha_i^j. \end{aligned}$$

Hence

$$\begin{aligned} \int \delta(\sigma_3(g_t)dv_{g_t}) &= \int \left[-6(\delta u)\sigma_3(g_t) - e^{-2u}\tilde{T}_{2i}^j \delta\alpha_j^i + n(\delta u)\sigma_3(g_t) \right] dv_{g_t} \\ &= \int \left[(n-6)(\delta u)\sigma_3(g_t) - e^{-2u}\tilde{T}_{2i}^j \delta\alpha_j^i \right] dv_{g_t}. \end{aligned} \quad (3.2)$$

On the other hand, the second term of (3.1) is

$$\begin{aligned}
& \int \frac{1}{3(n-4)} \delta(\tilde{P}_i^j \tilde{B}_j^i dv_{g_t}) \\
&= \int \frac{1}{3(n-4)} \left[\delta \tilde{P}_i^j \tilde{B}_j^i + \tilde{P}_i^j \delta \tilde{B}_j^i + n(\delta u) \tilde{P}_i^j \tilde{B}_j^i \right] dv_{g_t} \\
&= \int \frac{1}{3(n-4)} \left[\tilde{B}_j^i \left((-2\delta u) \tilde{P}_i^j - e^{-2u} (\delta \alpha_i^j) \right) + \tilde{P}_i^j \left(-4(\delta u) \tilde{B}_j^i + (n-4)e^{-4u} ((\delta u)^k (g^{il} C_{ljk} + C_{jlk}) \right. \right. \\
&\quad \left. \left. + 2(\delta u)^k u^l g^{ip} W_{pkjl}) \right) + n(\delta u) \tilde{P}_i^j \tilde{B}_j^i \right] dv_{g_t} \\
&= \int \frac{1}{3(n-4)} \left[(n-6)(\delta u) \tilde{P}_i^j \tilde{B}_j^i - e^{-2u} \tilde{B}_j^i (\delta \alpha_i^j) \right. \\
&\quad \left. + 2(n-4)e^{-4u} \tilde{P}_i^j \left((\delta u)^k g^{il} C_{ljk} + (\delta u)^k u^l g^{ip} W_{pkjl} \right) \right] dv_{g_t}
\end{aligned} \tag{3.3}$$

From calculations in (3.2) and (3.3), we have the following formula, which will be used in section 4

$$\begin{aligned}
& \delta(v^{(6)}(g_t) dv_{g_t}) \\
&= -\frac{1}{8} \left[(n-6)(\delta u) \sigma_3(g_t) - e^{-2u} \tilde{T}_{2i}^j (\delta \alpha_j^i) \right] dv_{g_t} - \frac{1}{24(n-4)} \left[(n-6)(\delta u) \tilde{P}_i^j \tilde{B}_j^i \right. \\
&\quad \left. - e^{-2u} \tilde{B}_j^i (\delta \alpha_j^i) + 2(n-4)e^{-4u} \tilde{P}_i^j \left((\delta u)^k g^{il} C_{ljk} + (\delta u)^k u^l g^{ip} W_{pkjl} \right) \right] dv_{g_t}.
\end{aligned} \tag{3.4}$$

Thus we have

$$\begin{aligned}
\delta F &= \int \delta(v^{(6)}(g_t) dv_{g_t}) \\
&= -\frac{1}{8} \int \left\{ (n-6)(\delta u) \sigma_3(g_t) - e^{-2u} \tilde{T}_{2i}^j \delta \alpha_j^i + \frac{1}{3(n-4)} \left[(n-6)(\delta u) \tilde{P}_i^j \tilde{B}_j^i \right. \right. \\
&\quad \left. \left. - e^{-2u} \tilde{B}_j^i (\delta \alpha_j^i) + 2(n-4)e^{-4u} \tilde{P}_i^j \left((\delta u)^k g^{il} C_{ljk} + (\delta u)^k u^l g^{ip} W_{pkjl} \right) \right] \right\} dv_{g_t} \\
&= \int \left\{ (n-6)(\delta u) v^{(6)}(g_t) - \frac{1}{8} \left[-e^{-2u} \tilde{T}_{2i}^j \delta \alpha_j^i - \frac{1}{3(n-4)} e^{-2u} \tilde{B}_j^i \delta \alpha_j^i \right. \right. \\
&\quad \left. \left. + \frac{2}{3} e^{-4u} \tilde{P}_i^j \left((\delta u)^k g^{il} C_{ljk} + (\delta u)^k u^l g^{ip} W_{pkjl} \right) \right] \right\} dv_{g_t}.
\end{aligned} \tag{3.5}$$

Proof of Theorem 1.1 Noting that $u(0) = 0$, we conclude the first variational formula of $\mathcal{F}_3[g_t]$ within the conformal class $[g]$ is (see [CF] or [G])

$$\begin{aligned}
\frac{d}{dt} \Big|_{t=0} \mathcal{F}_3(g_t) &= \frac{d}{dt} \Big|_{t=0} F \cdot V^{-\frac{n-6}{n}} - \frac{n-6}{n} V^{-\frac{n-6}{n}} \left(\int v^{(6)} dv \right) \int n \phi dv \\
&= V^{-\frac{n-6}{n}} \left\{ (n-6) \int \phi v^{(6)}(g) + \frac{1}{8} \int T_{2ij} \phi_{ij} + \frac{1}{8} \int \frac{B_{ij} \phi_{ij}}{3(n-4)} - \frac{1}{12} \int P_{ij} \phi_k C_{ijk} \right. \\
&\quad \left. - (n-6) V^{-1} \left(\int v^{(6)}(g) \right) \int \phi dv \right\} \\
&= (n-6) V^{-\frac{n-6}{n}} \left\{ \int \phi \left(v^{(6)} - V^{-1} \int v^{(6)} \right) dv \right\}.
\end{aligned} \tag{3.6}$$

where we have used (2.3) and (2) of Lemma 2.1 and the integration by parts. Here $V = \int dv_g$. Hence, we see that the Euler-Langrange equation of the functional $\mathcal{F}_3(g)$ within the conformal class $[g]$ is

$$v^{(6)}(g) = V^{-1} \int v^{(6)}(g) dv_g \equiv \text{const},$$

and we get Theorem 1.1. \square

4 The Second Variational Formula and proofs of Theorem 1.2-1.3

In this section, we will calculate the second variational formula for the functional \mathcal{F}_3 within the conformal class $[g]$. The computation is direct and routine. For convenience, we separate each term in the first variational equation (3.5) and compute them respectively.

For derivative of the first term in (3.5), by use of (3.4), we have

$$\begin{aligned} & (n-6) \frac{d}{dt} \Big|_{t=0} \int (\delta u) v^{(6)}(g_t) dv_{g_t} \\ &= (n-6) \int \left\{ \psi v^{(6)}(g) - \frac{1}{8} (\delta u) \left[(n-6) (\delta u) \sigma_3(g_t) - e^{-2u} \tilde{T}_{2i}^j \delta \alpha_j^i \right] dv_{g_t} \Big|_{t=0} \right. \\ & \quad - \frac{(\delta u)}{24(n-4)} \left[(n-6) (\delta u) \tilde{P}_i^j \tilde{B}_j^i - e^{-2u} \tilde{B}_j^i \delta \alpha_i^j \right. \\ & \quad \left. \left. + 2(n-4) e^{-4u} \tilde{P}_i^j \left((\delta u)^k g^{il} C_{ljk} + (\delta u)^k u^l g^{ip} W_{pkjl} \right) \right] dv_{g_t} \Big|_{t=0} \right\} \\ &= (n-6) \int \left\{ \psi v^{(6)}(g) + (n-6) \phi^2 v^{(6)}(g) - \frac{1}{8} \left[-\phi T_{2ij} \phi_{ij} \right. \right. \\ & \quad \left. \left. - \frac{\phi B_{ij} \phi_{ij}}{3(n-4)} + \frac{2}{3} \phi \phi_k P_{ij} C_{ijk} \right] \right\} dv. \end{aligned} \tag{4.1}$$

For derivative of the second term in (3.5), we need the following formula of the variation of the Newton transformation:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} (\tilde{T}_{2i}^j) &= \frac{d}{dt} \Big|_{t=0} \left(\frac{1}{2!} \delta_{i_1 i_2 i}^{j_1 j_2 j} \tilde{P}_{j_1}^{i_1} \tilde{P}_{j_2}^{i_2} \right) = \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{j_1}^{i_1} \frac{d}{dt} \Big|_{t=0} \tilde{P}_{j_2}^{i_2} \\ &= \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{i_1 j_1} (-2\phi P_{i_2 j_2} - \phi_{i_2 j_2}) = -4\phi T_{2ij} - \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{i_1 j_1} \phi_{i_2 j_2}. \end{aligned} \tag{4.2}$$

Therefore, the variation of the second term of (3.5) is given by

$$\begin{aligned} & \frac{1}{8} \frac{d}{dt} \Big|_{t=0} \int e^{-2u} \tilde{T}_{2i}^j \delta \alpha_j^i dv_{g_t} = \frac{1}{8} \int \left\{ -2\phi T_{2ij} \phi_{ij} + \frac{d}{dt} \Big|_{t=0} \tilde{T}_{2i}^j \phi_j^i \right. \\ & \quad \left. + T_{2ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} + n\phi T_{2ij} \phi_{ij} \right\} dv \\ &= \frac{1}{8} \int \left\{ (n-2) \phi T_{2ij} \phi_{ij} + T_{2ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} + \phi_{ij} \left(-4\phi T_{2ij} - \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{i_1 j_1} \phi_{i_2 j_2} \right) \right\} dv \\ &= \int \left\{ \frac{(n-6)}{8} \phi T_{2ij} \phi_{ij} + \frac{1}{8} T_{2ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} - \frac{1}{8} \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{i_1 j_1} \phi_{i_2 j_2} \phi_{ij} \right\}. \end{aligned} \tag{4.3}$$

The variation of the third term of (3.5) is

$$\begin{aligned}
& \frac{1}{24(n-4)} \frac{d}{dt} \Big|_{t=0} \int \left(e^{-2u} \tilde{B}_i^j \delta \alpha_j^i \right) dv_{g_t} \\
&= \frac{1}{24(n-4)} \int \left\{ -2\phi B_{ij} \phi_{ij} + \frac{d}{dt} \Big|_{t=0} (\tilde{B}_i^j) \phi_{ij} + B_{ij} \frac{d}{dt} \Big|_{t=0} \delta \alpha_{ij} + n\phi B_{ij} \phi_{ij} \right\} dv \\
&= \frac{1}{24(n-4)} \int \left\{ (n-2)\phi B_{ij} \phi_{ij} + B_{ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} + \phi_{ij} \left(-4\phi B_{ij} + 2(n-4)\phi_k C_{ijk} \right) \right\} dv \\
&= \int \left\{ \frac{n-6}{24(n-4)} \phi B_{ij} \phi_{ij} + \frac{B_{ij}}{24(n-4)} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} + \frac{1}{12} \phi_k \phi_{ij} C_{ijk} \right\} dv.
\end{aligned} \tag{4.4}$$

The variation of the fourth term of (3.5) is

$$\begin{aligned}
& -\frac{1}{12} \frac{d}{dt} \Big|_{t=0} \int \left[e^{-4u} \tilde{P}_j^i \left((\delta u)^k g^{jl} C_{ilk} + (\delta u)^k u^l g^{jp} W_{ikpl} \right) \right] dv_{g_t} \\
&= -\frac{1}{12} \int \left\{ -4\phi P_{ij} \phi_k C_{ijk} + \frac{d}{dt} \Big|_{t=0} \tilde{P}_i^j \phi^k g^{il} C_{ljk} + P_{ij} \psi_k C_{ijk} + P_{ij} \phi_k \phi_l W_{ikjl} + n\phi P_{ij} \phi_k C_{ijk} \right\} dv \\
&= -\frac{1}{12} \int \left\{ (n-4)\phi P_{ij} \phi_k C_{ijk} + P_{ij} \phi_k \phi_l W_{ikjl} + \phi_k C_{ijk} \left(-2\phi P_{ij} - \phi_{ij} \right) + P_{ij} \psi_k C_{ijk} \right\} dv \\
&= \int \left\{ -\frac{(n-6)}{12} \phi P_{ij} \phi_k C_{ijk} - \frac{1}{12} P_{ij} \phi_k \phi_l W_{ikjl} + \frac{1}{12} \phi_k \phi_{ij} C_{ijk} - \frac{1}{12} P_{ij} \psi_k C_{ijk} \right\} dv.
\end{aligned} \tag{4.5}$$

Combining (4.1), (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
& \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) \\
&= \int \left\{ (n-6)\psi v^{(6)}(g) + (n-6)\phi^2 v^{(6)}(g) + \frac{n-6}{4} \phi T_{2ij} \phi_{ij} - \frac{1}{8} \left[-\frac{2(n-6)\phi B_{ij} \phi_{ij}}{3(n-4)} \right. \right. \\
&\quad \left. \left. + \frac{4(n-6)}{3} \phi \phi_k P_{ij} C_{ijk} - T_{2ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} - \frac{B_{ij}}{3(n-4)} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} \right. \right. \\
&\quad \left. \left. + \delta_{i_1 i_2 i}^{j_1 j_2 j} P_{i_1 j_1} \phi_{i_2 j_2} \phi_{ij} - \frac{4}{3} \phi_k \phi_{ij} C_{ijk} + \frac{2}{3} P_{ij} \phi_k \phi_l W_{ikjl} + \frac{2}{3} P_{ij} \psi_k C_{ijk} \right] \right\} dv.
\end{aligned} \tag{4.6}$$

Since $\frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} = \psi_{ij} - 2\phi_i \phi_j + |\nabla \phi|^2 g_{ij}$, by use of divergence theorem we obtain

$$\begin{aligned}
& -\int T_{2ij} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} - \int \frac{B_{ij}}{3(n-4)} \frac{d^2}{dt^2} \Big|_{t=0} \alpha_{ij} \\
&= -\int T_{2ij} (\psi_{ij} - 2\phi_i \phi_j + |\nabla \phi|^2 g_{ij}) - \int \frac{B_{ij} (\psi_{ij} - 2\phi_i \phi_j + |\nabla \phi|^2 g_{ij})}{3(n-4)} \\
&= \int -\frac{2}{3} P_{kl} C_{kli} \psi_i + 2T_{2ij} \phi_i \phi_j + \frac{2}{3(n-4)} B_{ij} \phi_i \phi_j - |\nabla \phi|^2 T_{2kk} \\
&= \int -\frac{2}{3} P_{kl} C_{kli} \psi_i - |\nabla \phi|^2 T_{2kk} - 2T_{2ij,j} \phi_i \phi - 2T_{2ij} \phi_{ij} \phi - \frac{2}{3(n-4)} B_{ij,j} \phi_i \phi - \frac{2}{3(n-4)} B_{ij} \phi_{ij} \phi \\
&= \int -\frac{2}{3} P_{kl} C_{kli} \psi_i - |\nabla \phi|^2 T_{2kk} + \frac{4}{3} \phi \phi_i P_{kl} C_{kli} - 2T_{2ij} \phi_{ij} \phi - \frac{2}{3(n-4)} B_{ij} \phi_{ij} \phi,
\end{aligned} \tag{4.7}$$

where we have used the following identity in the second equality

$$\int T_{2ij}\psi_{ij}dv + \frac{1}{3(n-4)} \int B_{ij}\psi_{ij}dv = \int \frac{2}{3}P_{kl}C_{kli}\psi_i,$$

which can be checked by use of (2.3), (2) of Lemma 2.1 and integration by parts.

Substituting (4.7) into (4.6) and making some cancelations, we conclude that

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} F(g_t) = \int \Big\{ & (n-6)\psi v^{(6)} + (n-6)^2\phi^2 v^{(6)}(g) - \frac{1}{8} \Big[-2(n-5)\phi T_{2ij}\phi_{ij} \\ & - \frac{2(n-5)}{3(n-4)}\phi B_{ij}\phi_{ij} + \frac{4(n-5)}{3}\phi P_{ij}\phi_k C_{ijk} \\ & - \frac{4}{3}\phi_k\phi_{ij}C_{ijk} + \delta_{kli}^{mnj}P_{km}\phi_{ln}\phi_{ij} + \frac{2}{3}P_{ij}\phi_k\phi_l W_{ikjl} - (n-2)|\nabla\phi|^2\sigma_2(g) \Big] \Big\} dv, \end{aligned} \quad (4.8)$$

where we have used the identity that $T_{2kk} = (n-2)\sigma_2(g)$ (see Lemma 2.2). It remains to study the last four terms on the right hand side of (4.8). By definition,

$$\begin{aligned} \delta_{kli}^{mnj} &= \det \begin{pmatrix} \delta_{km} & \delta_{kn} & \delta_{kj} \\ \delta_{lm} & \delta_{ln} & \delta_{lj} \\ \delta_{im} & \delta_{in} & \delta_{ij} \end{pmatrix} \\ &= \delta_{km}\delta_{ln}\delta_{ij} - \delta_{km}\delta_{lj}\delta_{in} - \delta_{lm}\delta_{kn}\delta_{in} + \delta_{lm}\delta_{in}\delta_{kj} + \delta_{im}\delta_{kn}\delta_{lj} - \delta_{im}\delta_{ln}\delta_{kj}. \end{aligned}$$

We compute by use of divergence theorem

$$\begin{aligned} \int \delta_{kli}^{mnj} P_{km}\phi_{ln}\phi_{ij} &= \int \phi \left(\delta_{kli}^{mnj} (P_{mk}\phi_{nl})_{,ij} \right) \\ &= \int \phi \delta_{kli}^{mnj} \left(P_{km,i} \phi_{nl,j} + P_{km,i} \phi_{nl,j} + P_{km}\phi_{nl,ij} \right). \end{aligned} \quad (4.9)$$

Now we compute integrands of the right hand side of (4.9) respectively. The first term is

$$\begin{aligned} \phi \delta_{kli}^{mnj} P_{km,i} \phi_{nl,j} &= \phi P_{km,i} \phi_{nl} (\delta_{km}\delta_{ln}\delta_{ij} - \delta_{km}\delta_{lj}\delta_{in} - \delta_{lm}\delta_{kn}\delta_{in} + \delta_{lm}\delta_{in}\delta_{kj} + \delta_{im}\delta_{kn}\delta_{lj} - \delta_{im}\delta_{ln}\delta_{kj}) \\ &= \phi P_{kk,ii}\phi_{nn} - \phi P_{kk,ij}\phi_{ij} - \phi P_{kl,ii}\phi_{kl} + \phi P_{kl,ik}\phi_{il} + \phi P_{ki,il}\phi_{kl} - \phi P_{ki,ik}\phi_{ll} \\ &= \phi \phi_{nn} C_{iik,k} + \phi \phi_{kl} C_{lik,i} + \phi \phi_{kl} C_{iki,l} = \phi \phi_{kl} C_{lik,i}. \end{aligned} \quad (4.10)$$

The second one is

$$\begin{aligned} 2\phi \delta_{kli}^{mnj} P_{km,i} \phi_{nl,j} &= 2\phi P_{kk,i}\phi_{nni} - 2\phi P_{kk,i}\phi_{ijj} - 2\phi P_{km,i}\phi_{kmi} + 2\phi P_{km,i}\phi_{imk} + 2\phi P_{km,m}\phi_{kll} - 2\phi P_{km,m}\phi_{llk} \\ &= 2\phi P_{km,i}(\phi_{mik} - \phi_{mki}) = 2\phi P_{km,i}\phi_j R_{jmik}. \end{aligned} \quad (4.11)$$

The third one is

$$\begin{aligned} \phi \delta_{kli}^{mnj} P_{km}\phi_{nl,ij} &= \phi \phi_{kk,ii} P_{nn} - \phi \phi_{iikl} P_{kl} - \phi \phi_{klii} P_{kl} + \phi \phi_{ilki} P_{kl} + \phi \phi_{kiil} P_{kl} - \phi \phi_{kii} P_{ll} \\ &= \phi P_{nn}(\phi_{kii} - \phi_{kii}) + \phi P_{kl}(\phi_{ilki} - \phi_{kii}) + \phi P_{kl}(\phi_{kiil} - \phi_{iikl}) \\ &= \phi P_{nn}(-\phi_{mik} R_{mk} - \phi_m R_{mk,k}) + \phi P_{kl}(\phi_{mi} R_{mlk} + \phi_m R_{ikml,i}) + \phi P_{kl}(\phi_{ml} R_{mk} + \phi_m R_{mk,l}) \end{aligned} \quad (4.12)$$

where we have used the Ricci identity in the last equality.

Substituting the following identities into (4.11) and (4.12),

$$\begin{aligned} R_{ij} &= (n-2)P_{ij} + \frac{R}{2(n-1)}g_{ij}, \quad R_{ij,i} = \frac{R_j}{2}, \quad P_{kk} = \frac{R}{2(n-1)}; \\ R_{ijkl} &= W_{ijkl} + P_{ik}g_{jl} + P_{jl}g_{ik} - P_{il}g_{jk} - P_{jk}g_{il}, \quad P_{kk,i} = P_{ik,k}; \\ R_{ikml,i} &= R_{kl,m} - R_{km,l} = (n-2)C_{klm} + \frac{\nabla_m R g_{kl} - \nabla_l R g_{km}}{2(n-1)}. \end{aligned}$$

after making some cancelations we see that the left hand side of (4.9) becomes

$$\begin{aligned} & \int \phi \delta_{kli}^{mnj} P_{km} \phi_{nl} \phi_{ij} \\ &= \int \left[-\phi \phi_{kl} C_{lki,i} + \frac{4-n}{2(n-1)} \phi R \phi_{mk} P_{mk} - \frac{R^2}{4(n-1)^2} \phi \Delta \phi - \frac{n-2}{4(n-1)^2} \phi R \phi_m R_{,m} \right. \\ & \quad + \phi P_{kl} \phi_{mi} W_{mlik} + \phi |P_{kl}|^2 \Delta \phi + (n-4) \phi P_{kl} \phi_{ml} P_{mk} + n \phi P_{kl} \phi_m P_{kl,m} \\ & \quad \left. + 2 \phi P_{km,i} \phi_j W_{jmik} - 2 \phi P_{km,i} \phi_k P_{mi} \right]. \end{aligned} \tag{4.13}$$

On the other hand, by divergence theorem, we see that the other three terms on the last of (4.8) are

$$\begin{aligned} \frac{2}{3} \int P_{ij} \phi_k \phi_l W_{ikjl} &= \frac{2}{3} \int -\phi P_{ij,k} \phi_l W_{ikjl} - \phi P_{ij} \phi_{kl} W_{ikjl} - \phi P_{ij} \phi_l W_{ikjl,k} \\ &= \int -\frac{2}{3} \phi P_{ij,k} \phi_l W_{ikjl} - \frac{2}{3} \phi P_{ij} \phi_{kl} W_{ikjl} - \frac{2(n-3)}{3} \phi P_{ij} \phi_l C_{ijl}, \end{aligned} \tag{4.14}$$

$$-\frac{4}{3} \int \phi_k \phi_{ij} C_{ijk} = \int \frac{4}{3} \phi \phi_{ijk} C_{ijk} + \frac{4}{3} \phi \phi_{ij} C_{ijk,k}, \tag{4.15}$$

$$\begin{aligned} & - \int (n-2) |\nabla \phi|^2 \sigma_2(g) \\ &= \int (n-2) \phi \Delta \phi \sigma_2(g) + (n-2) \phi \phi_i (\sigma_2(g))_{,i} \\ &= \int \frac{(n-2)}{2} \phi \Delta \phi \left(\frac{R^2}{4(n-1)^2} - |P_{kl}|^2 \right) + (n-2) \phi \phi_i \left(\frac{R R_{,i}}{4(n-1)^2} - P_{kl} P_{kl,i} \right) \\ &= \int \frac{(n-2) \phi \Delta \phi R^2}{8(n-1)^2} - \frac{n-2}{2} \phi \Delta \phi |P_{kl}|^2 + \frac{(n-2) \phi R \phi_i R_{,i}}{4(n-1)^2} - (n-2) \phi \phi_i P_{kl} P_{kl,i}. \end{aligned} \tag{4.16}$$

By combining equations (4.13), (4.14), (4.15) and (4.16) and doing some cancelations, we conclude that the last four terms on the right hand side of (4.8) are equal to

$$\begin{aligned} & -\frac{1}{8} \int \left[\frac{1}{3} \phi \phi_{kl} B_{kl} + (n-4) \phi T_{2ij} \phi_{ij} - \frac{2(n-6)}{3} \phi P_{ij} \phi_k C_{ijk} \right. \\ & \quad \left. + \frac{4}{3} \phi C_{kmi} \phi_j W_{jmik} + \frac{4}{3} \phi \phi_{ijk} C_{ijk} \right] dv, \end{aligned} \tag{4.17}$$

where we have used $T_{2ij} = \sigma_2(g)\delta_{ij} - \sigma_1(g)P_{ij} + P_{ik}P_{kj}$ and $B_{ij} = C_{ijk,k} + P_{kl}W_{ikjl}$. Moreover,

$$\begin{aligned}
& \frac{4}{3}\phi C_{ijk}\phi_{jik} \\
&= \frac{4}{3}\phi C_{ijk}(\phi_{jki} + \phi_m R_{mjik}) \\
&= \frac{4}{3}\phi C_{ijk}\phi_m(W_{mjik} + P_{mi}g_{jk} + P_{jk}g_{mi} - P_{mk}g_{ij} - P_{ij}g_{mk}) \\
&= \frac{4}{3}\phi C_{ijk}\phi_m W_{mjik} + \frac{4}{3}\phi C_{ijj}\phi_m P_{mi} + \frac{4}{3}\phi C_{ijk}\phi_i P_{jk} - \frac{4}{3}\phi C_{iik}\phi_m P_{mk} - \frac{4}{3}\phi C_{ijk}P_{ij}\phi_k \\
&= -\frac{4}{3}\phi\phi_m C_{ijk}W_{mjki} - \frac{4}{3}\phi C_{ijk}P_{ij}\phi_k,
\end{aligned}$$

where we used $\sum_i C_{iik} = 0$ and $C_{ijk} = -C_{ikj}$.

Thus it follows that (4.17) is equal to

$$-\frac{1}{8} \int \left[\frac{1}{3}\phi\phi_{kl}B_{kl} + (n-4)\phi\phi_{ij}T_{2ij} - \frac{2(n-6)}{3}\phi\phi_k P_{ij}C_{ijk} - \frac{4}{3}\phi\phi_k C_{ijk}P_{ij} \right] dv. \quad (4.18)$$

Substituting (4.18) into (4.8), we conclude that

$$\begin{aligned}
& \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) \quad (4.19) \\
&= \int \left\{ (n-6)\psi v^{(6)} + (n-6)^2\phi^2 v^{(6)}(g) - \frac{1}{8} \left[-2(n-5)\phi\phi_{ij}T_{2ij} - \frac{2(n-5)}{3(n-4)}\phi\phi_{ij}B_{ij} \right. \right. \\
&\quad \left. \left. + \frac{4(n-5)}{3}\phi\phi_k P_{ij}C_{ijk} + \frac{1}{3}\phi\phi_{kl}B_{kl} + (n-4)\phi\phi_{ij}T_{2ij} - \frac{2(n-6)}{3}\phi\phi_k P_{ij}C_{ijk} - \frac{4}{3}\phi\phi_k C_{ijk}P_{ij} \right] \right\} \\
&= \int \left\{ (n-6)\psi v^{(6)} + (n-6)^2\phi^2 v^{(6)} + \frac{1}{8} \left[(n-6)\phi\phi_{ij}T_{2ij} + \frac{n-6}{3(n-4)}\phi\phi_{ij}B_{ij} \right. \right. \\
&\quad \left. \left. - \frac{2(n-6)}{3}\phi\phi_k P_{ij}C_{ijk} \right] \right\}.
\end{aligned}$$

Proof of Theorem 1.2 By Theorem 1.1, at the critical metric of the functional $\mathcal{F}_3(g)$, it holds that $v^{(6)}(g)$ should be constant, and it follows that $F(g) = Vv^{(6)}(g)$. By our notations $\mathcal{F}_3[g_t] = \frac{F(g_t)}{(\int dv_{g_t})^{(n-6)/n}}$. By use of (4.19) and (3.6), at the critical metric g , we have

$$\begin{aligned}
& \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3[g_t] \quad (4.20) \\
&= \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) \cdot V^{-\frac{n-6}{n}} + 2 \frac{d}{dt} \Big|_{t=0} F(g_t) \cdot \frac{d}{dt} \Big|_{t=0} V(g_t)^{-\frac{n-6}{n}} + F(g) \frac{d^2}{dt^2} \Big|_{t=0} V(g_t)^{-\frac{n-6}{n}} \\
&= \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) \cdot V^{-\frac{n-6}{n}} - \frac{2(n-6)}{n} \left(\frac{d}{dt} \Big|_{t=0} F(g_t) \right) \cdot V^{-\frac{(2n-6)}{n}} \int n\phi \\
&\quad + F \left\{ (n-6)(2n-6)V^{-\frac{(3n-6)}{n}} \left(\int \phi \right)^2 - n(n-6)V^{-\frac{(2n-6)}{n}} \int \phi^2 - (n-6)V^{-\frac{(2n-6)}{n}} \int \psi \right\} \\
&= V^{-\frac{n-6}{n}} \left\{ \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) - 2(n-6)^2 v^{(6)}(g) V^{-1} \left(\int \phi \right)^2 + (n-6)(2n-6) v^{(6)}(g) V^{-1} \left(\int \phi \right)^2 \right. \\
&\quad \left. - n(n-6) v^{(6)}(g) \int \phi^2 - (n-6) v^{(6)}(g) \int \psi \right\}
\end{aligned}$$

$$\begin{aligned}
&= V^{-\frac{n-6}{n}} \left\{ \frac{d^2}{dt^2} \Big|_{t=0} F(g_t) + 6(n-6)v^{(6)}(g)V^{-1} \left(\int \phi \right)^2 - n(n-6)v^{(6)}(g) \int \phi^2 - (n-6)v^{(6)}(g) \int \psi \right\} \\
&= V^{-\frac{n-6}{n}} \left\{ \int \left[-6v^{(6)}(g) \left(\phi - V^{-1} \int \phi \right)^2 + \frac{\phi\phi_{kl}}{24(n-4)} B_{kl} + \frac{1}{8} \phi\phi_{mk} T_{2mk} \right. \right. \\
&\quad \left. \left. - \frac{1}{12} \phi C_{ijk} P_{ij} \phi_k \right] dv \right\}.
\end{aligned}$$

If we define an operator \mathcal{L} by

$$\mathcal{L}(f) := \frac{B_{ij}f_{ij}}{24(n-4)} + \frac{1}{8}T_{2ij}f_{ij} - \frac{1}{12}P_{ij}C_{ijk}f_k,$$

for $f \in C^\infty(M)$. It is easy to see that \mathcal{L} is self-adjoint with respect to the L^2 inner product of the Riemannian manifold. Indeed, for any two smooth functions f and h , we have

$$\begin{aligned}
\langle \mathcal{L}(f), h \rangle &= \int_M \mathcal{L}(f) h dv \\
&= \int_M \left[\frac{B_{ij}f_{ij}h}{24(n-4)} + \frac{1}{8}T_{2ij}hf_{ij} - \frac{1}{12}P_{ij}C_{ijk}f_k h \right] dv \\
&= \int_M \left[-\frac{B_{ij}f_i h_j}{24(n-4)} - \frac{1}{8}T_{2ij}f_i h_j - \frac{B_{ij,j}f_i h}{24(n-4)} - \frac{1}{8}T_{2ij,j}f_i h - \frac{1}{12}P_{ij}C_{ijk}f_k h \right] dv \\
&= \int_M \left[-\frac{B_{ij}f_i h_j}{24(n-4)} - \frac{T_{2ij}f_i h_j}{8} \right] dv \\
&= \langle f, \mathcal{L}(h) \rangle,
\end{aligned}$$

where we have used (2) and (3) in Lemma 2.1, (2.3) and integration by parts. Denote $\phi - V^{-1} \int \phi$ by $\bar{\phi}$. From (4.20), we see that

$$\begin{aligned}
&\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3(g_t) \\
&= (n-6)V^{-\frac{n-6}{n}} \left\{ \int \left[-6v^{(6)}(g)\bar{\phi}^2 + \mathcal{L}(\phi)\phi \right] dv \right\} \\
&= (n-6)V^{-\frac{n-6}{n}} \left\{ \int \left[-6v^{(6)}(g)\bar{\phi}^2 + \mathcal{L}(\bar{\phi})\bar{\phi} \right] dv \right\}.
\end{aligned} \tag{4.21}$$

Thus we complete the proof of Theorem 1.2. □

To prove Theorem 1.3, we need the following famous theorem.

Theorem 4.1 (Lichnerowicz and Obata, see e.g. [L]). *Let M be an n -dimensional compact manifold. Suppose the Ricci curvature of M is bounded from below by*

$$\text{Ric} \geq (n-1)K$$

for some positive constant K , then the first nonzero eigenvalue of the Laplacian on M must satisfy

$$\lambda_1 \geq nK.$$

Moreover, equality holds if and only if M is isometric to a standard sphere of radius $\frac{1}{\sqrt{K}}$.

By the min-max principle, for the first nonzero eigenvalue λ_1 of Laplacian, it holds that

$$\lambda_1 \int_M f^2 dv \leq \int_M |\nabla f|^2 dv, \quad (4.22)$$

for any $f \in C^\infty(M)$ satisfying $\int_M f dv = 0$.

Proof of Theorem 1.3. Note that an Einstein manifold (M^n, g) is a critical metric in $[g]$, i.e. it satisfies (1.2). Now let (M^n, g) be an Einstein manifold with positive scalar curvature, then it follows from Theorem 4.1 and (4.22) that

$$\frac{R}{n-1} \int_M \bar{\phi}^2 dv \leq \int_M |\nabla \bar{\phi}|^2 dv. \quad (4.23)$$

Note that for an Einstein manifold, $v^{(6)}(g) = -\frac{(n-2)R^3}{386n^2(n-1)^2}$, $\mathcal{L}(\phi) = \frac{(n-2)R^2}{64n^2(n-1)}\Delta\phi$. Hence, we see that

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3[g_t] &= (n-6)V^{-\frac{n-6}{n}} \int_M \left[\frac{(n-2)R^3}{64n^2(n-1)^2} \bar{\phi}^2 + \frac{(n-2)R^2\bar{\phi}}{64n^2(n-1)} \Delta\bar{\phi} \right] dv, \\ &= \frac{(n-2)(n-6)R^2}{64n^2(n-1)} V^{-\frac{n-6}{n}} \int_M \left[\frac{R\bar{\phi}^2}{n-1} - |\nabla \bar{\phi}|^2 \right] dv \\ &\leq \frac{(n-2)(n-6)R^2}{64n^2(n-1)} V^{-\frac{n-6}{n}} \left(\frac{R}{n-1} - \lambda_1 \right) \int_M \bar{\phi}^2 dv \\ &\leq 0, \end{aligned} \quad (4.24)$$

with equality holds if and only if $\lambda_1 = \frac{R}{n-1}$. Hence, by Theorem 4.1, in this case (M, g) is isometric to the standard sphere S^n .

Therefore, we prove that an Einstein manifold with positive scalar curvature must be a strict local maximum “point” within its conformal class $[g]$ unless (M, g) is isometric to S^n with a multiple of the standard metric. We complete the proof of theorem 1.3. \square

Remark 4.1. Let (M^n, g) be an n -dimensional Einstein manifold with nonpositive scalar curvature, then we have from the proof of Theorem 1.3 (see (4.24))

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3(g_t) \leq 0,$$

that is, it is stable.

Remark 4.2. When M^n is an Einstein manifold with positive scalar curvature with dimension $n = 5$, we see from (4.24) that

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}_3[g_t] \geq 0, \quad (4.25)$$

with equality if and only if $\lambda_1 = \frac{R}{4}$. Theorem 4.1 shows that in this case (M^5, g) is isometric to the sphere S^5 with the standard metric up to a multiple of constant. And we see that this Einstein metric is a strict local minimum of the functional \mathcal{F}_3 within its conformal class if the equality does not hold in (4.25).

Remark 4.3. Let $\mathcal{T}_{ij}(g) = T_{2ij}(g) + \frac{1}{n-4}(B_g)_{ij}$, we have

$$\sum_j \nabla^j \mathcal{T}_{ij} = 0,$$

that is, \mathcal{T}_{ij} is a divergence-free tensor. We observe that $v^{(6)}(g) = -24 \sum_{ij} \mathcal{T}_{ij}(g)(P_g)_{ij}$.

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